Optimal Iterative Learning Control Design for Continuous-Time Systems with Nonidentical Trial Lengths Using Alternating Projections Between Multiple Sets

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Abstract

Iterative learning control (ILC) applies to systems that repeat the same finite duration task repeatedly. Each repetition is usually termed as a trial, and the associated duration is called the trial length. Once a trial is completed, all information is available for use in updating the control input for the subsequent trial. The vast majority of the currently available designs demand a strictly identical trial length. This paper gives a new result on the design and analysis for continuous-time linear dynamics based on a modified alternating projection method, where the trial lengths may be nonidentical. This result employs multiple sets to represent the actual varying trial length dynamics and is developed by reformulating the problem to one that minimizes the defined distance in a Hilbert space setting. Compared to the standard alternating projections using two sets, the theory of alternating projections between multiple sets is employed to obtain deterministic convergence result for the nonidentical trial length problem. A numerical case study is also given to illustrate the application of the new design.

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1. Introduction

To illustrate the basics of iterative learning control (ILC), consider a robot executing a pick-and-place task. The operations required are: i) collect the object or payload from a fixed location, ii) transfer it over a finite duration, iii) place the payload onto a moving conveyor, iv) return to the starting location, and then repeat this sequence as many times as required or until a hat is needed for maintenance or other reasons. Once a trial is complete, all information is available for use in updating the control input for the subsequent trial.

Let $y_k(t)$, $0 \leq \alpha < \infty, k \geq 0$, and denotes an ILC variable, where y is the vector or scalar valued variable of interest, α is the trial length, and $k \geq 0$ is the trial number. Suppose also that a reference vector or trajectory is specified over the trial length, where in the pick-and-place robot example this would be the desired path for the robot to follow between the pick and place locations. Then, the error on this trial is the difference between the reference and the output.

Given the error sequence, the ILC design problem can be formulated as constructing a sequence of trial inputs that will force this error, as measured by some suitable norm, either to zero or to within some suitable tolerance. The first work is widely credited to [1] and sources for the early literature include the survey papers [2, 3, 4]. Application areas include robotic-assisted biomedical/rehabilitation devices, see, e.g., [5], multi-agent systems [6, 7], batch processing [8], and motion control systems [9, 10].

A significant proportion of the available ILC designs assume that each trial is of the same length. However, there are examples where this is not the case, and one example is the ventricular assist device considered in [11]. In this application, a rotary blood pump needs to be controlled repeatedly to maintain the blood perfusion, but the cycle duration may not be uniform as the heart rate varies when the patient stays in different situations, e.g., exercise or rest. Similar issues exist in other ILC applications including foot motion and lower limb movement where ILC is used to regulate the assistive functional electrical stimulation [12]. For safety reasons, the stimulation signal should be ended up once initial contact is detected between the foot and the ground, which also leads to the nonidentical trial length problem.

If the theory for constant trial ILC is not applicable, it is necessary to develop algorithms for variable trial length ILC design. A high order ILC design, termed trial averaging ILC, was developed in [13] to solve the ILC design problem for randomly varying trial length examples. However, the learning efficiency will decrease as the trials increase due to the use of redundant past tracking information. Even though two improved versions of the trial averaging scheme were developed in [14] to reduce the redundant learning from much early historical information, the most recent previous trial is still a more efficient choice [15]. Nonetheless, results based on trial averaging have been considered further in [16, 17] as the robustness to the variation of trial lengths can become stronger by employing the trial averaging method. However, trial averaging usually requires the use of actual trial information. In contrast, correct predictions based on the absent outputs may also achieve better learning efficiency. An auxiliary predictive model has been developed in [18] to compensate for the unavailable output data, which gives another direction for solving the nonidentical trial length problem. Moreover, a model free adaptive ILC for randomly varying trial lengths was reported in [19], which only uses the system input and output data. The lifted ILC framework for discrete dynamics, also known as intermittent ILC [20], has been employed to achieve P-type ILC design for the nonidentical trial length problem [12, 21].

The aforementioned results mainly conduct the design and analysis in stochastic sense. In contrast, a deterministic model was built in [22] for the nonidentical trial length problem. Different from the stochastic model that requires the trial lengths randomly varying, the deterministic model gives an iteration-dependent assumption on nonidentical trial lengths, i.e., there always exists at least one trial reaching to the desired lengths during a fixed successive trials. When considering continuous-time systems, this assumption is more realistic in practice because the actual trial length cannot reach to every existing lengths during the learning process. Note that as long as the full-length trial occurs infinitely many times, a well learning process can be guaranteed along the trial axis. Therefore, we build a deterministic model for the nonidentical trial length problem in this paper and the strong convergence result of optimal ILC designs for continuous-time systems is given. Moreover, existing design and analysis techniques for the nonidentical trial length problem mainly includes the conventional contraction mapping method [13, 14] and the Lyapunov-based composite energy function method [23, 24]. However, both methods can be abstracted as mappings between defined metric spaces. If each trial can be seen as an point in a suitable space, e.g., a Hilbert space, then the complex design and analysis problem can be simplified by using the language of operator theory [25]. Introducing an intuitive and simple operator-based design and analysis technique for the nonidentical trial length problem is the main motivation of this paper.

Based on the operator theory, the alternating projection (also successive projection) method can further transfer the ILC problem into an intuitive projection problem in the defined Hilbert space. Therefore, the standard alternating projection method has been developed and applied to, among others, design for input constraints [26], point-to-point [27], and spatial path tracking [28, 29] problems. However, this method is not naturally applicable to the nonidentical trial length problem since a single convex set cannot represent the varying trial length dynamics. On the other hand, it is desired to develop new effective design and analysis framework to deal with systems with nonidentical trial lengths. Therefore, this paper modifies the alternating projection framework by employing multiple sets to represent the system dynamics, and strong convergence result of alternating projections between multiple sets is considered. It should be emphasized that multiple sets are introduced only to represent the varying trial length dynamics. The learning process will not be influenced if reasonable projection orders are designed.

Moreover, optimization is naturally used as the paradigm for design and analysis under alternating projections, while there has been some optimal ILC results dealing with the nonidentical trial length problem. A norm optimal ILC application for nonidentical trial lengths was explored in [11], but there is no strict convergence analysis for this extension. An optimal learning control scheme, which still uses the most recent available information, was proposed in [30] by defining a specific cost function accordingly. By recursively computing the variable learning gain along the time axis, this scheme can thus obtain faster convergence speed. Also, an intermittent optimal ILC was reported based on alternating projections for discrete-time systems with nonidentical trial lengths in [31]. However, this previous work is based on the standard alternating projections, namely, two closed sets are employed and hence only the convergence result in random sense is obtained. To achieve stronger convergence result, the standard alternating projection can be modified by introducing multiple sets to deal with the nonidentical trial lengths. Note also that aforementioned optimal ILC designs can only be applied to continuous-time dynamics after first applying sampling, whereas the results in this paper can be applied directly to continuous-time dynamics. This is also an advantage of the alternating projection method that uses the Hilbert space settings.

In what follows, an optimal ILC design framework for continuous-time systems with nonidentical trial lengths by alternating projections between multiple sets is developed. By employing an auxiliary subspace as the tracking objective, the ILC design problem with nonidentical trial lengths can be transformed into finding a point in the intersection region of multiple closed affine subspaces, then a projection sequence converging in norm is developed by defining a projecting order between the auxiliary subspace and a family of closed affine subspaces. This designed projection sequence can be implemented by the norm optimal ILC with specific modifications for the nonidentical trial length case. Also, an accelerated ILC scheme is developed. The implementation of the accelerated scheme is developed on the basis of the modified norm optimal ILC. Finally, the effectiveness of the optimal ILC designs, including comparative aspects, is demonstrated by a simulation case study using a multiple-input multiple-output (MIMO) numerical model. The new contributions of this paper are:

- An optimal ILC design framework is developed to solve the nonidentical trial length problem based on alternating projections between multiple sets.
- Using this setting, the norm optimal ILC is specifically modified for the nonidentical trial length problem, and its strong convergence result is strictly proved.
- An accelerated scheme of the modified norm optimal ILC for the nonidentical trial length case is developed under this alternating projection framework, whose convergence analysis is also given accordingly.

The structure is organized as follows: the problem description and the background to alternating projections between multiple sets are developed in Section 2. Section 3 develops an optimal ILC design for the nonidentical trial length problem using alternating projections, and specifically modified the norm optimal ILC applied. Section 4 develops the accelerated ILC scheme and details its implementation. A numerical case study is given in Section 5 and Section 6 gives the conclusion and discusses possible future research.

Notation: $\mathbb{N}$ and $\mathbb{N}_+$ represent the set of natural numbers and positive integers, respectively. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ represent the sets of n-dimensional real vectors and $n\times m$ real matrices, respectively. $L_2^m[a,b]$ represents the Lebesgue 2-space of $\mathbb{R}^m$ valued signals on an interval $[a,b]$. The superscript T and $\perp$ respectively represent the transpose and the orthogonal complement space and $x\perp y$ denotes orthogonal vectors x and y. $\bigcap$ represents the intersection of sets. $\langle\cdot\rangle$ represents the inner product. $\mathbb{X}\times\mathbb{Y}$ represents the Cartesian product of two spaces $\mathbb{X}$ and $\mathbb{Y}$. Other notations will be introduced as required.

2. Preliminaries

Consider a continuous-time linear time-invariant MIMO system in the ILC setting with nonidentical trial lengths

$$\begin{aligned} \dot{x}_k(t) &= Ax_k(t) + Bu_k(t), \\ y_k(t) &= Cx_k(t), \end{aligned} \tag{1}$$

where $t \in [0, T_k] \subset \mathbb{R}$ and the subscript $k \in \mathbb{N}$ represent time and trial number, respectively. Also, T_k denotes the trial length for trial k which is unknown until the trial ends. Moreover, $x_k(t) \in \mathbb{R}^n$, $u_k(t) \in \mathbb{R}^\ell$ and $y_k(t) \in \mathbb{R}^m$ respectively represent the system state, control input and output vectors on trial k. Finally, it is assumed that $x_k(0) = x_0$ for all trials.

Let $y_d(t) \in \mathbb{R}^m$ for $t \in [0, T]$ denote the reference trajectory vector. In this work, it is assumed that $T_k < T$, where T denotes the desired trial length. Also let $T_- > 0$ denote the minimum trial length, then $T_k \in [T_-, T]$. For analysis, the trial length is modified to enable analysis based on a trial length that is the same for each trial as discussed next, i.e.,

$$y_k(t) = \begin{cases} y_k(t), & t \in [0, T_k], \\ y_d(t), & t \in (T_k, T]. \end{cases} \tag{2}$$

In effect, for $t > T_k$, the pre-specified reference trajectory, is used to obtain signals defined over the same trial length. Also, define the input and output spaces as $L_2^\ell[0, T]$ and $L_2^m[0, T]$ and the desired trajectory vector $y_d \in L_2^m[0, T]$. Moreover, the effects of the state initial conditions, as in [32], can be incorporated into the corresponding signals in a way such that they can be taken as zero in the subsequent analysis. The system dynamics (1) can now be written in the operator form

$$y_k = F_{j_k} G u_k, \tag{3}$$

where $u_k \in L_2^\ell[0, T]$ and $y_k \in L_2^m[0, T]$ and the convolution operator $G : L_2^\ell[0, T] \to L_2^m[0, T]$ takes the form

$$(Gu_k)(t) = \int_0^t C\mathbf{e}^{A(t-\tau)} B u_k(\tau)\, d\tau. \tag{4}$$

$F_{j_k} : L_2^m[0, T] \to L_2^m[0, T]$ is a linear operator that modifies the output signals in (2) and, for an arbitrary $\zeta \in L_2^m[0, T]$ takes the form

$$F_{j_k}\zeta = \begin{cases} \zeta(t), & t \in [0, T_k], \\ y_d(t), & t \in (T_k, T]. \end{cases} \tag{5}$$

Finally, the trial error is denoted by e_k, and $e_k = y_d - y_k$.

In this paper, an alternating projection in a Hilbert space is used to solve the ILC design problem, for which the required background is given next.

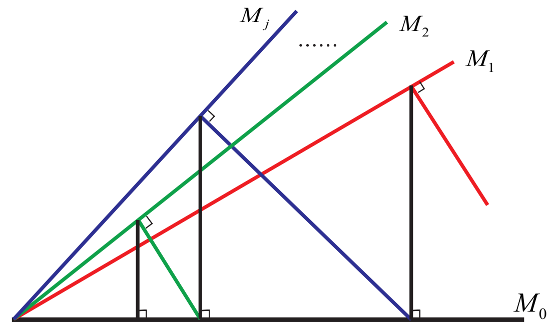

Figure 1: Illustration of the alternating projections between multiple affine subspaces.

2.1. Alternating projection interpretation

As a branch of the optimal ILC, the standard alternating projection method in ILC [26, 27], usually employs two convex sets, which can only correspond to fixed trial length ILC problems for convergence in norm. The results in this paper require alternating projections between more than two sets and some modifications are required relative to the previous work. An intuitive illustration is shown in Fig. 1, where different colors represent projections on different affine subspaces.

To begin with, a family of closed affine subspaces M_j in a Hilbert space are introduced to represent the system dynamics with nonidentical trial lengths, and another closed subspace M_0, also in a Hilbert space, is used to represent the ILC tracking objective. These sets are defined as

$$M_j = \{(e,u) \in H : e = y_d - y,\ y = F_j G u\}, \tag{6}$$

$$M_0 = \{(e,u) \in H : e = 0\}, \tag{7}$$

where j is an index taking values in $\mathbb{N}_+$, and H is a Hilbert space defined as

$$L_2^m[0,T] \times L_2^\ell[0,T], \tag{8}$$

whose inner product and associated induced norm are

$$\langle (e,u),(y,v)\rangle_{\{Q,R\}} = \int_0^T e^T(t)\,Qy(t)\,dt + \int_0^T u^T(t)\,Rv(t)\,dt, \tag{9}$$

$$\|(e,u)\|_{\{Q,R\}} = \sqrt{\langle (e,u),(e,u)\rangle_{\{Q,R\}}}. \tag{10}$$

In the above formulation, Q and R are symmetric positive definite weighing matrices with compatible dimensions, and $y \in L_2^m[0,T]$ and $v \in L_2^\ell[0,T]$. Also

$F_j : L_2^m[0,T] \to L_2^m[0,T]$ is defined by

$$F_j\zeta = \begin{cases} \zeta(t), & t \in \left[0, T^j\right], \\ y_d(t), & t \in \left(T^j, T\right], \end{cases} \tag{11}$$

where the superscript j on variables, e.g. T^j, denotes members of the family of affine subspaces, and for this purpose, the following definition is also required.

Definition 1. *To define the order of the affine subspaces with respect to the index j, the followings are used:*

- $T^1 = T_-$*, and* $F_1\zeta = \begin{cases} \zeta(t), & t \in [0, T_-], \\ y_d(t), & t \in (T_-, T]; \end{cases}$
- $T^\infty = T$*, and* $F_\infty\zeta = \zeta$*;*
- *For any* $a, b \in \mathbb{N}_+$*, if* $a < b$*, then* $T^a < T^b$*.*

Given Definition 1, M_j, for $j \in \mathbb{N}_+$, has an order corresponding to the positive integer sets $\mathbb{N}_+$, which also corresponds to the trial lengths of actual outputs ordered from small to large. Given that there actually exist an infinite number of affine subspaces, the one corresponding to the desired length, i.e. M_∞, has to correspond to $j = \infty$ by the second entry in Definition 1. This, in turn, establishes that

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M_\infty. \tag{12}$$

Remark 1. *The relationship (12) reveals a deterministic property of the nonidentical trial length problem, which is abstracted in the settings of Hilbert space for analysis. To further explain, for instance,* $M_1 \subseteq M_\infty$ *means the event that the actual trial length reaches to the minimum trial length* T_- *is the premise of event that it reaches to the desired one* T*. Based on this deterministic property, it will be shown later in this paper that (12) is a necessary condition for the convergence of alternating projections between an infinite number of affine subspaces.*

To ensure that the ILC problem considered has a solution, it is required that at least one point exists in the intersection region of M_0 and M_j, for $j \in \mathbb{N}_+$.

Assumption 1. *There exists a point* $(0, u^*)$ *in the intersection region of the multiple subspaces defined in (6) and (7), i.e.,* $(0, u^*) \in \bigcap_0^\infty M_j$*.*

With above settings, the ILC problem is equivalent to find a point belonging to the intersection region of multiple sets defined in (6) and (7). In this sense, the ILC design objective is to construct a projection point sequence, denoted by $\{z_k = (e_k, u_k)\}_{k\geq 0}$, to ensure the process of alternating projections converges to a point belonging to the intersection region.

3. ILC design using alternating projections

In this section, an ILC design for nonidentical trial length problem is developed using the alternating projections setup given in the previous section.

3.1. Design for converging alternating projections

Firstly, a projection point sequence is defined with respect to the sequence $\{j_k\}_{k\geq 0}$ as

$$z_{k+1} = P_{j_{k+1}}(z_k), \; k \geq 0, \tag{13}$$

where $P_{j_{k+1}}$ represents a projection operator that projects a signal onto $M_{j_{k+1}}$. If all M_j are closed subspaces instead of closed affine subspaces, there has been result showing convergence of alternating projections between a finite number of M_j. Based on $\{z_k\}_{k\geq 0}$, the following result holds, whose proofs can be found in [33] and as Theorem 4.4 in [34].

Lemma 1. *If the sequence $s = \{j_k\}_{k\geq 0}$ takes every value in $\{1, 2, \ldots, J\}$ infinitely many times, i.e.*

$$\Delta(s, i) = \sup_k \left[K_{k+1}(i) - K_k(i)\right] < \infty, \tag{14}$$

for each $i \in \{1, 2, \ldots, J\}$, and there exists a constant S, only associated with the sequence $\{j_k\}_{k\geq 0}$, such that

$$\|z_q - z_p\|^2 \leq S \sum_{k=p}^{q-1} \|z_{k+1} - z_k\|^2, \quad p, q \in \mathbb{N}_+, \quad q > p \geq 1, \tag{15}$$

then $\{z_k\}_{k\geq 0}$ converges in norm to the orthogonal projection of z_0 onto $\bigcap_1^J M_j$, where $J \in \mathbb{N}_+$ and $\{K_k(i)\}_{k\geq 0}$ is an increasing integer sequence such that $j_{K_k(i)} = i$ with $K_0(i) = 0$.

The condition (14) in Lemma 1 requires that each i, for $i \in \{1, 2, \ldots, J\}$, occurs infinitely many times in sequence $\{j_k\}_{k\geq 0}$. More formally, the difference of the index numbers between two successive appearances of the same i should be bounded. In the considered problem, it seems to require that each trial length should be reached infinitely many times, while this is not possible in applications of continuous-time systems. To overcome this difficulty, the condition (14) can be specifically simplified for the nonidentical trial length case. Recall that M_∞ is defined as the affine subspace corresponding to the desired trial length, for which a less conservative assumption specifically for the considered problem can be made in contrast to the requirements in Lemma 1.

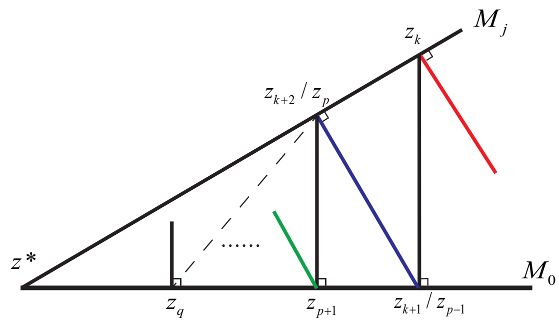

Figure 2: Geometric illustration of alternating projections between M_0 and M_j.

Assumption 2. *M_∞ appears infinitely many times in the alternating projection process, i.e.*

$$\Delta(s,\infty) = \sup_k \left[K_{k+1}(\infty) - K_k(\infty)\right] < \infty. \tag{16}$$

Remark 2. *In Assumption 2, M_∞ represents the subspace corresponding to the desired trial length, which means the desired trial length should appear infinitely many times. This assumption is reasonable because there is not enough information for learning if the actual trial length cannot reach the desired one sufficiently many times. A similar assumption with respect to the desired length, termed the persistent full-learning property, is given in [22], where it is required that the desired trial length should be reached after every fixed number of successive trials. This paper gives another description of deterministic model for the nonidentical trial length case using the modified alternating projections.*

Lemma 1 requires that alternating projections should be between finite number of subspaces, i.e., J subspaces, while an infinite number of subspaces should be considered for the continuous-time setting of this paper. Nonetheless, strong convergence result can still be guaranteed based on a deterministic property of the nonidentical trial length problem, i.e., the condition (12). To show that alternating projections between an infinite number of subspaces can still converge under (12), a geometric illustration of alternating projections between M_0 and M_j, is given in Fig. 2, which also gives an intuitive illustration of the convergence analysis of sequence $\{z_k\}_{k\geq 0}$. The following theorem can now be established.

Theorem 1. *If the projecting order of alternating projections between multiple subspaces defined in (6) and (7) satisfies*

$$M_{j_k} = \begin{cases} M_j, & k \text{ odd}, \\ M_0, & k \text{ even}, \end{cases} \tag{17}$$

then the sequence $\{z_k\}_{k\geq 0}$ *converges in norm to a point belonging to* $\bigcap_0^\infty M_j$.

Proof. By Assumption 1, there exists a point z^* belonging to the intersection region $\bigcap_0^\infty M_j$. Based on the idempotency and self-adjointness of an orthogonal projection operator, it follows that

$$\begin{aligned}\langle z - P_j(z), P_j(z) - z^*\rangle &= \langle z - z^*, P_j(z) - z^*\rangle + \langle z^* - P_j(z), P_j(z) - z^*\rangle \\ &= \langle z, P_j(z)\rangle - \langle z, z^*\rangle + \langle z^*, P_j(z)\rangle - \langle P_j(z), P_j(z)\rangle \\ &= \langle z^*, P_j(z)\rangle - \langle z, z^*\rangle = \langle P_j(z^*), z\rangle - \langle z, z^*\rangle = 0,\end{aligned} \tag{18}$$

and hence

$$\begin{aligned}\|z - P_j(z)\|^2 &= \|z - P_j(z) + P_j(z) - z^*\|^2 - \|P_j(z) - z^*\|^2 \\ &\quad - 2\langle z - P_j(z), P_j(z) - z^*\rangle \\ &= \|z - z^*\|^2 - \|P_j(z) - z^*\|^2.\end{aligned} \tag{19}$$

Adding the trial number k gives

$$\|z_k - z^*\|^2 - \|z_{k+1} - z^*\|^2 = \|z_k - z_{k+1}\|^2, \tag{20}$$

and hence yields

$$\|z_p - z^*\|^2 - \|z_q - z^*\|^2 = \sum_{k=p}^{q-1} \|z_{k+1} - z_k\|^2. \tag{21}$$

Next a property for any $q > p \geq 1$ is firstly proved. To begin with, when p is odd and q is even for $q > p \geq 2$, as shown in Fig. 2, it follows that

$$z_q = z_{p-1} + \gamma (z_{p+1} - z_{p-1}), \tag{22}$$

where γ is a scalar. Given (20), $\|z_k - z^*\|^2$ monotonically decreases as k increases, so $z_q \in M_0$ should be a point on the line segment with endpoints z_{p+1} and z^*. Then

$$\|z_q - z_{p-1}\|^2 = \gamma^2 \|z_{p+1} - z_{p-1}\|^2 \geq \|z_{p+1} - z_{p-1}\|^2, \tag{23}$$

which yields $\gamma \geq 1$. When z_q converges to z^*,

$$\langle z_q - z_p, z_{p-1} - z_p\rangle = 0. \tag{24}$$

Also, it follows from $\langle z_{p+1} - z_{p-1}, z_p - z_{p+1} \rangle = 0$ that

$$\gamma = -\frac{\|z_p - z_{p-1}\|^2}{\langle z_{p+1} - z_{p-1}, z_{p-1} - z_p \rangle} = \frac{\|z_p - z_{p-1}\|^2}{\langle z_{p+1} - z_{p-1}, z_p - z_{p+1} + z_{p+1} - z_{p-1} \rangle} \tag{25}$$

$$= \frac{\|z_p - z_{p-1}\|^2}{\|z_{p+1} - z_{p-1}\|^2}, \tag{26}$$

and hence $1 \leq \gamma \leq \frac{\|z_p - z_{p-1}\|^2}{\|z_{p+1} - z_{p-1}\|^2}$. Given (19), since $\langle z_p - z_{p+1}, z_{p+1} - z^* \rangle = 0$ and $\langle z_p - z_{p-1}, z_p - z^* \rangle = 0$, it follows that

$$\begin{aligned}
\langle z_p - z_q, z^* - z_q \rangle &= \langle (z_p - z_{p+1}) + (z_{p+1} - z_q), z^* - z_{p-1} - \gamma (z_{p+1} - z_{p-1}) \rangle \\
&= \langle z_p - z_{p+1}, (1-\gamma)(z^* - z_{p-1}) \rangle + \langle z_p - z_{p+1}, \gamma (z^* - z_{p+1}) \rangle \\
&+ \langle (1-\gamma)(z_{p+1} - z_{p-1}), (z^* - z_{p-1}) - \gamma (z_{p+1} - z_{p-1}) \rangle \\
&= \langle z_p - z_{p-1} + z_{p-1}, (1-\gamma)(z^* - z_{p-1}) \rangle - (1-\gamma) \langle z_{p+1}, z^* - z_{p-1} \rangle \\
&+ (1-\gamma) \langle z_{p+1}, z^* - z_{p-1} \rangle - (1-\gamma) \langle z_{p-1}, z^* - z_{p-1} \rangle \\
&- (1-\gamma) \gamma \|z_{p+1} - z_{p-1}\|^2 \\
&= (1-\gamma) \left(\langle z_p - z_{p-1}, z^* - z_{p-1} \rangle - \gamma \|z_{p+1} - z_{p-1}\|^2 \right),
\end{aligned} \tag{27}$$

which is a quadratic function with respect to γ and its quadratic coefficient is positive. Moreover, when $\gamma = \frac{\|z_p - z_{p-1}\|^2}{\|z_{p+1} - z_{p-1}\|^2}$,

$$\begin{aligned}
\langle z_p - z_{p-1}, z^* - z_{p-1} \rangle - \gamma \|z_{p+1} - z_{p-1}\|^2 &= \langle z_p - z_{p-1}, z^* - z_{p-1} \rangle - \langle z_p - z_{p-1}, z_p - z_{p-1} \rangle \\
&= \langle z_p - z_{p-1}, z^* - z_p \rangle = 0.
\end{aligned} \tag{28}$$

Hence $\langle z_p - z_q, z^* - z_q \rangle \leq 0$ since $1 \leq \gamma \leq \frac{\|z_p - z_{p-1}\|^2}{\|z_{p+1} - z_{p-1}\|^2}$. For the case $p = 1$, the original orthogonal projection point of z_1 belonging to M_0 can be employed and the same result obtained. Then

$$\begin{aligned}
\|z_q - z_p\|^2 &= \|z_p - z^*\|^2 - \|z_q - z^*\|^2 + 2 \langle z_p - z_q, z^* - z_q \rangle \\
&\leq \|z_p - z^*\|^2 - \|z_q - z^*\|^2, \ q > p \geq 1.
\end{aligned} \tag{29}$$

When p is even and q is odd, a similar result to (29) is obtained by employing two points in the same subspaces that z_q belongs to, i.e., M_{j_q}, where one point is the orthogonal projection of z_p onto M_{j_q}, and the another is the original orthogonal projection point of z_p. Furthermore, when both p and q are odd or even, (29) is obtained since $\|z_k - z^*\|^2$ monotonically decreases as k increases by (21) even though multiple subspaces arise when k is odd. The final step is to establish the property (29) for any $q > p \geq 1$, where, combined with (21), it follows that

$$\|z_q - z_p\|^2 \leq \|z_p - z^*\|^2 - \|z_q - z^*\|^2 = \sum_{k=p}^{q-1} \|z_{k+1} - z_k\|^2, \tag{30}$$

and then $S = 1$ as required by Lemma 1. Hence the sequence $\{z_k\}_{k\geq 0}$ would converge in norm to a point belonging to $\bigcap_0^\infty M_j$ as established next.

The proof is divided into two parts, the first of which is to prove that $\{z_k\}_{k\geq 0}$ converges in norm in a Hilbert space and the second is to prove that the convergent point, denoted by z_∞, belongs to $\bigcap_0^\infty M_j$. For the first part, it follows from (20) that $\|z_k - z^*\|^2$ monotonically decreases as k increases and is bounded below by 0. Therefore, there exists a constant $\beta > 0$ such that $\lim\limits_{k\to\infty} \|z_k - z^*\|^2 = \beta$. Furthermore, given $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $0 \leq \|z_p - z^*\|^2 - \beta < \epsilon/2$ whenever $p \geq k$ and this is the case when $q \geq k$. Combined with (29), it follows that

$$\|z_q - z_p\|^2 \leq \|z_p - z^*\|^2 - \beta + \beta - \|z_q - z^*\|^2 < \epsilon/2 + \epsilon/2 = \epsilon. \tag{31}$$

Then, due to the completeness property of Hilbert spaces, $\{z_k\}_{k\geq 0}$ converges in norm to a point, i.e., z_∞.

To prove the second part, note that M_∞ appears infinitely many times as Assumption 2 requires. Hence there exists a sub-sequence $\left\{z_{K_k(\infty)}\right\}_{k\geq 0}$ such that each $z_{K_k(\infty)} \in M_\infty$, and then

$$\left\langle z_{K_k(\infty)}, z' \right\rangle = 0, \tag{32}$$

for every $z' \in M_\infty^\perp$. By the continuity property of the inner product, it follows from (32) that

$$\langle z_\infty, z' \rangle = \left\langle \lim_{k\to\infty} z_{K_k(\infty)}, z' \right\rangle = \lim_{k\to\infty} \left\langle z_{K_k(\infty)}, z' \right\rangle = 0, \tag{33}$$

and hence z_∞ is orthogonal to every point in $M_\infty^\perp$ and thus $z_\infty \in M_\infty$. Note also that M_0 appears infinitely many times with $\Delta(s, 0) = 2$, because of the designed order (17). Hence $z_\infty \in M_0$ and $z_\infty \in M_\infty \cap M_0$, where

$$M_\infty \cap M_0 = \{(0, u_\infty) \in H : y_d = F_\infty G u_\infty = G u_\infty\}. \tag{34}$$

Substituting u_∞ into M_j gives

$$e = y_d - F_j G u_\infty = y_d - F_j y_d = 0, \tag{35}$$

and hence $(0, u_\infty) \in M_j$ for any $j \in \mathbb{N}_+$. Finally, $z_\infty \in \bigcap_0^\infty M_j$ and the proof is complete. $\square$

Remark 3. *Lemma 1 actually discussed the alternating projections between multiple closed subspaces as in [33], which means the original point naturally belongs to the intersection region. In contrast, due to the system dynamics setting of the nonidentical trial length problem, Theorem 1 should analyze the convergence of the alternating projections between a family of closed affine subspaces and a subspace, where the original point would not belong to the intersection region. Nevertheless, the projection sequence* $\{z_k\}_{k\geq 0}$ *still converges to a point belonging to the intersection region as in Theorem 1.*

There is no direct relationship between z^* and the convergent point z_∞, except that both belong to $\bigcap_0^\infty M_j$. Theorem 1 gives a design of projecting order for convergence, when the alternating projection method is employed to solve the nonidentical trial length problem. However, the design can be modified to further improve the performance of ILC design subject to the condition that the convergence property is guaranteed. The following corollary to Theorem 1 can now be established.

Corollary 1. *The convergent point* z_∞ *is the orthogonal projection of* z_0 *onto* $\bigcap_0^\infty M_j$.

Proof. Note that z^* is a point that belongs to $\bigcap_0^\infty M_j$, and therefore $z \in M_{j_{k+1}}$. Since $z_k - P_{j_{k+1}}(z_k) \in M_{j_{k+1}}^\perp$, it follows that

$$\langle z_k - z_{k+1}, z^* \rangle = \left\langle z_k - P_{j_{k+1}}(z_k), z^* \right\rangle = 0, \tag{36}$$

and from (36) that

$$\begin{aligned} \langle z_0 - z_\infty, z^* \rangle &= \lim_{k\to\infty} \langle z_0 - z_k, z^* \rangle \\ &= \lim_{k\to\infty} \left(\langle z_0 - z_1, z^* \rangle + \langle z_1 - z_2, z^* \rangle + \cdots + \langle z_{k-1} - z_k, z^* \rangle \right) = 0. \end{aligned} \tag{37}$$

Hence $z_0 - z_\infty \in \left(\bigcap_0^\infty M_j\right)^\perp$ and therefore

$$z_0 = \overbrace{z_\infty}^{\in \bigcap_0^\infty M_j} + \overbrace{z_0 - z_\infty}^{\in \left(\bigcap_0^\infty M_j\right)^\perp}. \tag{38}$$

Consequently, z_∞ is the orthogonal projection of z_0 onto $\bigcap_0^\infty M_j$ by the projection theorem for Hilbert spaces. □

Corollary 1 establishes that the ILC design can not only enforce the sequence to converge, but also to the orthogonal projection point of the initial point onto the intersection region. This is a critical property, especially when there is more than one point in the intersection region. Moreover, when an arbitrary initial ILC input u_0 is chosen, the ILC design will always converge with minimum distance with respect to u_0 in the defined Hilbert space provided Theorem 1 holds.

3.2. ILC implementation of projections

For applications, an implementation of the optimal ILC design is required. This problem is actually equivalent to minimizing a cost function that is associated with the distance in the defined Hilbert space. According to the inner product and associated induced norm defined in (9) and (10), the cost function is given as

$$J\left(u_{k+1}\right) = \left\|e_{k+1}\right\|_{Q}^{2} + \left\|u_{k+1} - u_{k}\right\|_{R}^{2}. \tag{39}$$

Remark 4. *Note that the output signal y_k is modified by adding the desired trajectory information as required, and hence the corresponding parts of e_k are set as zero. Although this zero compensation mechanism cannot be of benefit to learning efficiency, it can be seen as a lazy pattern [13, 35]. In this sense, the learning efficiency is only dependent on the actual trial length that each trial would reach. Moreover, the output error signals under zero compensation mechanism can also accurately show the actual situation of control process and there would be no influence of the virtual data. Hence the convergence property of the ILC design under alternating projections can be analyzed exactly.*

Minimizing the cost function (39) is an optimal ILC problem and many approaches can be employed to develop a solution. Norm optimal ILC has many advantages both in theory and practical applications, see, e.g., [25], and can be used to solve this optimal ILC problem. Note that the tracking error vector, i.e., $e_k = y_d - F_{j_k} G u_k$ according to (3), has been modified for the nonidentical trial length case, then the standard form of the solution to the norm optimal ILC problem can be modified as follows:

$$u_{k+1} = u_k + G^{*}(I + GG^{*})^{-1} e_k, \tag{40}$$

where G^* denotes the adjoint operator of G in Hilbert space and I denotes the unit operator in this space. The update law (40) is actually non-causal, while it can be implemented in a simple feedforward form according to the cost function or a causal feedback plus feedforward structure by solving a series of differential equations [36, 37]. It should be emphasized that this work mainly aims at proposing an ILC design and analysis framework for continuous-time systems in the presence of nonidentical trial lengths. In this sense, strict convergence analysis of the norm optimal ILC applicable to the nonidentical trial length case can be conducted. The following result is now relevant.

Proposition 1. *The input sequence $\{u_k\}_{k\geq 0}$ generated by the update law (40) converges to u_∞, where $z_\infty = (0, u_\infty)$.*

Proof. Since the update law (40) is one of the solutions that minimize the cost function (39), the points consisting of input and output signals generated by (3) and (40), denoted as $\tilde{z} = (\tilde{e}, \tilde{u})$, should be the orthogonal projections onto M_j, i.e., $\tilde{z} = (\tilde{e}, \tilde{u}) \in M_j$. Also, denote $z = (e, u)$ as the corresponding orthogonal projection point onto M_0. Then

$$\begin{aligned} P_0(\tilde{z}) &= \arg\min_{\hat{z}\in M_0} \|\hat{z} - \tilde{z}\|_H^2 \\ &= \arg\min_{(\hat{e},\hat{u})\in M_0} \|(\hat{e},\hat{u}) - (\tilde{e},\tilde{u})\|_{\{Q,R\}}^2 \\ &= \arg\min_{(\hat{e},\hat{u})\in M_0} \left\{ \|\hat{e} - \tilde{e}\|_Q^2 + \|\hat{u} - \tilde{u}\|_R^2 \right\}, \end{aligned} \tag{41}$$

whose solution is $\hat{u} = \tilde{u}$ because $(\hat{e}, \hat{u}) \in M_0$. Hence $\hat{e} = 0$, and

$$\begin{aligned} P_j(z) &= \arg\min_{\hat{z}\in M_j} \|\hat{z} - z\|_H^2 \\ &= \arg\min_{(\hat{e},\hat{u})\in M_j} \|(\hat{e},\hat{u}) - (e,u)\|_{\{Q,R\}}^2 \\ &= \arg\min_{(\hat{e},\hat{u})\in M_j} \left\{ \|\hat{e} - 0\|_Q^2 + \|\hat{u} - u\|_R^2 \right\} \\ &= \arg\min_{\hat{u}} \left\{ \|\hat{e}\|_Q^2 + \|\hat{u} - u\|_R^2 \right\}, \end{aligned} \tag{42}$$

which is an optimization problem that can be solved by (40). Then, repeatedly utilizing (40) to solve the optimization problem in (42) as k increases is equivalent to conducting alternating projections under the order given by (17). Finally, the sequence $\{u_k\}_{k\geq 0}$ generated by (40) converges to u_∞ by Theorem 1 and the proof is complete. □

It follows from Proposition 1 that the input signal u_k is only updated when projecting on M_j under the designed order (17), which means that two projections in the sequence $\{z_k\}_{k\geq 0}$ corresponds to one iteration in the design. Using Proposition 1, convergence properties of ILC design under alternating projections for the non-identical trial length problem can be further investigated, which gives rise to the following theorem.

Theorem 2. *Given system (1), applying the ILC update law (40) under the designed order of alternating projections (17) with initial input signal u_0 yields the zero convergence property of the output errors, i.e.,*

$$\lim_{k\to\infty} \|e_k\| = 0, \tag{43}$$

and

$$\|u_k - u_\infty\| \geq \|u_{k+1} - u_\infty\|. \tag{44}$$

Proof. By Theorem 1, the sequence $\{z_k\}_{k\geq 0}$ converges to z_∞ under the designed order (17), so the distance between z_k and z_∞ in the defined Hilbert space converges to zero, i.e.,

$$\lim_{k\to\infty} \|z_k - z_\infty\| = \lim_{k\to\infty} \left\{ \|e_k - 0\|_Q^2 + \|u_k - u_\infty\|_R^2 \right\} = 0, \tag{45}$$

and the zero convergence property (43) is established. Furthermore, it follows from $\langle z_k - z_{k+1}, z_{k+1} - z_\infty \rangle = 0$ that

$$\|z_k - z_\infty\|^2 = \|z_k - z_{k+1}\|^2 + \|z_{k+1} - z_\infty\|^2 + 2\langle z_k - z_{k+1}, z_{k+1} - z_\infty \rangle \geq \|z_{k+1} - z_\infty\|^2. \tag{46}$$

Hence when k is even

$$\|u_k - u_\infty\|_R^2 \geq \|e_{k+1}\|_Q^2 + \|u_{k+1} - u_\infty\|_R^2, \tag{47}$$

which gives rise to (44) and the proof is complete. □

It is shown in Theorem 2 that the projections with specific implementations under the designed order of Theorem 1 can solve the nonidentical trial length problem in ILC. Although this projection implementation is an optimal ILC scheme, the monotonic property with respect to the error signal cannot be deduced because the affine subspaces representing the system dynamics are unknown. Equivalently, the monotonic property of error signal cannot be naturally obtained under the design using alternating projection method because of the nonidentical trial lengths. To be more specific, the relationship between two successive trial errors under the modified norm optimal ILC (40) is

$$e_{k+1} = y_d - F_{j_{k+1}} G u_{k+1} = \left(y_d - F_{j_{k+1}} G u_k\right) - F_{j_{k+1}} G G^* \left(I + G G^*\right)^{-1} e_k. \tag{48}$$

Note that $e_k \neq y_d - F_{j_{k+1}} G u_k$, since typically $F_{j_{k+1}} \neq F_{j_k}$. Then, a direct equality relation between e_{k+1} and e_k cannot be derived from (48), which means the monotonic convergence of error in norm cannot directly obtained. This can also be interpreted under the proposed alternating projection framework. For a certain angle between M_j and M_0 (certain weighting matrices Q and R), the deviation in norm of trial lengths between two successive trials, i.e. projecting onto two successive different M_j, may be larger, on the scale, than the distance that is decreasing, especially when z_k comes nearly close to z_∞.

However, the monotonic convergence result in norm can actually be achieved by modifying the error with virtual information, see, e.g. [12]. To further reveal the convergence of alternating projections in this paper, we extend the analysis in [12] to the continuous-time systems with nonidentical trial lengths under the modified norm optimal ILC. To begin with, define a full-length error $\bar{e}_k \in L_2^m[0,T]$ as

$$\bar{e}_k = y_d - Gu_k, \tag{49}$$

which consists of both actual error information e_k and virtual information generated by the full-length input u_k. Define a cutting operator $F_{j_k}^c : L_2^m[0,T] \to L_2^m[0,T]$ as

$$F_{j_k}^c \zeta = \begin{cases} \zeta(t), & t \in [0, T_k], \\ 0, & t \in (T_k, T]. \end{cases} \tag{50}$$

Then, a monotonic convergence result is given by the following proposition.

Proposition 2. *Given system (1), if choose symmetric positive definite weighing matrices satisfying*

$$\left\| I - GG^* (I + GG^*)^{-1} F_j^c \right\| < 1, \forall j \in \mathbb{N}_+, \tag{51}$$

then applying the modified norm optimal ILC (40) yields

$$\|\bar{e}_{k+1}\| \leq \|\bar{e}_k\|. \tag{52}$$

Proof. Combined with (40), it is easy to show that

$$\bar{e}_{k+1} = y_d - Gu_k - GG^* (I + GG^*)^{-1} e_k = \left[I - GG^* (I + GG^*)^{-1} F_{j_k}^c \right] \bar{e}_k. \tag{53}$$

Then, taking norm on the both side of (53) in the defined topology space, i.e. the Hilbert space H, (52) can be achieved when (51) is satisfied. □

This monotonic convergence result in norm is reasonable when considering the full-length error $\bar{e}_k$, which is equivalent to the case with identical trial lengths in essence except that the learning process is only fully stored in the full-length input signal instead of the error. Then, the proposed optimal ILC scheme can be further accelerated using alternating projections by modifications to the projecting method as established in the next section.

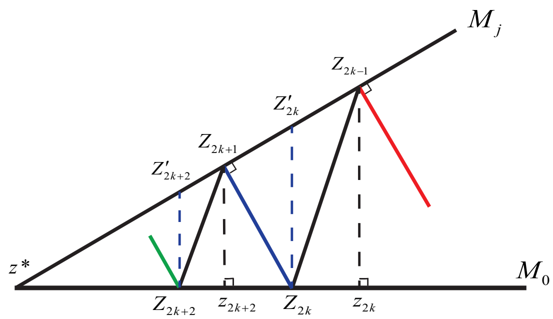

Figure 3: Illustration of the accelerated ILC scheme.

4. Accelerated ILC scheme

The projection developed in the previous section is actually not the optimal solution of ILC problem, and an accelerated version of the norm optimal ILC has been investigated for the constant trial length case [32]. In particular, a point closer to the point z^* always exists in the corresponding subspace compared to the orthogonal projection point. The aim of this section is to use the modified alternating projection framework to design and analyze the accelerated scheme for the nonidentical trial length case. An illustration of the accelerated ILC scheme is given in Fig. 3, where points that follow the solid lines usually have the accelerated convergence property compared to that of the dashed lines when projecting onto M_0. Then, an accelerated ILC scheme for the nonidentical trial length problem is first developed.

By Proposition 1, the control input signal only updates when projecting on M_j, which means M_0 is actually an auxiliary subspace for optimal ILC design under alternating projections. Equivalently, the sub-sequence $\{z_{2k} \in M_0\}_{k\geq 0}$ has no direct influence on the practical ILC process when following the designed order (17) in Theorem 1. Therefore, some accelerated strategies can be embedded into the process of finding a point in M_0. In this sense, a point closer to z^*, denoted as $Z = (0, U)$, can be located by the orthogonal projection point onto M_0 and employed to accelerate the ILC design.

This construction is illustrated in Fig. 3, and the sub-sequence of the orthogonal projection point $\{z_{2k} \in M_0\}_{k\geq 0}$ is employed to locate the closer point Z_{2k} when a point in M_0 is found. In contrast, the orthogonal projection is still conducted when we find a point in M_j. In this way, choose $z_0 \in H$ arbitrarily and set $Z_0 = P_0(z_0)$, then an accelerated strategy can be obtained by designing a sequence $\{Z_k\}_{k\geq 0}$, i.e.,

$$Z_{2k+1} = P_{j_{2k+1}}\left(Z_{2k}\right), \tag{54}$$
$$z_{2k+2} = P_{j_{2k+2}}\left(Z_{2k+1}\right), \tag{55}$$
$$Z_{2k+2} = Z_{2k} + \gamma_{2k}\left(z_{2k+2} - Z_{2k}\right), \tag{56}$$

where γ_{2k} is a scalar and

$$M_{j_k} = \begin{cases} M_j, & k \text{ odd}, \\ M_0, & k \text{ even}. \end{cases}$$

Compared with the sequence $\{z_k\}_{k\geq 0}$ designed in Theorem 1, $\{Z_k\}_{k\geq 0}$ changes the point belonging to M_0 to a point closer to z^* so as to accelerate the process of alternating projections. However, there should exist some limitations on this closer point to avoid the divergence. Then, in order to ensure the convergence of accelerated ILC design, the following theorem is established.

Theorem 3. *If the accelerated factor, γ_{2k}, satisfies*

$$1 \leq \gamma_{2k} \leq \frac{\left\|Z_{2k+1} - Z_{2k}\right\|^2}{\left\|z_{2k+2} - Z_{2k}\right\|^2}, \tag{57}$$

then the sequence $\{Z_k\}_{k\geq 0}$ converges to a point in $\bigcap_0^\infty M_j$.

Proof. Firstly, note that Z_{2k+1} and z_{2k+2} are the orthogonal projections of Z_{2k} and Z_{2k+1} respectively, and it follows from (18) that $\langle Z_{2k} - Z_{2k+1}, Z_{2k+1} - z^*\rangle = 0$ and $\langle Z_{2k+1} - z_{2k+2}, z_{2k+2} - z^*\rangle = 0$. Then, similar to (27),

$$\langle Z_{2k+1} - Z_{2k+2}, z^* - Z_{2k+2}\rangle = (1-\gamma_{2k})\left(\langle Z_{2k+1} - Z_{2k}, z^* - Z_{2k}\rangle - \gamma_{2k}\left\|z_{2k+2} - Z_{2k}\right\|^2\right). \tag{58}$$

Due to (57), $\langle Z_{2k+1} - Z_{2k+2}, z^* - Z_{2k+2}\rangle \leq 0$ by Theorem 1. Moreover, it follows that

$$\begin{aligned} \left\|Z_{2k+1} - z^*\right\|^2 &= \left\|Z_{2k+1} - Z_{2k+2}\right\|^2 + \left\|Z_{2k+2} - z^*\right\|^2 + 2\left\langle Z_{2k+1} - Z_{2k+2}, Z_{2k+2} - z^*\right\rangle \\ &\geq \left\|Z_{2k+1} - Z_{2k+2}\right\|^2 + \left\|Z_{2k+2} - z^*\right\|^2, \end{aligned} \tag{59}$$

and also

$$\begin{aligned} \left\|Z_{2k} - z^*\right\|^2 &= \left\|Z_{2k} - Z_{2k+1}\right\|^2 + \left\|Z_{2k+1} - z^*\right\|^2 + 2\left\langle Z_{2k} - Z_{2k+1}, Z_{2k+1} - z^*\right\rangle \\ &= \left\|Z_{2k} - Z_{2k+1}\right\|^2 + \left\|Z_{2k+1} - z^*\right\|^2, \end{aligned} \tag{60}$$

due to the orthogonal projection. Combined with (59) and (60), it follows that

$$\|Z_k - Z_{k+1}\|^2 \leq \|Z_k - z^*\|^2 - \|Z_{k+1} - z^*\|^2. \tag{61}$$

Finally, following the proof of two parts in Theorem 1 gives that $\{Z_k\}_{k\geq 0}$ converges to Z_∞, where Z_∞ is a point belonging to $\bigcap_0^\infty M_j$. □

One difference between the orthogonal projection sequence $\{z_k\}_{k\geq 0}$ and its accelerated design $\{Z_k\}_{k\geq 0}$ is that the scalar γ employed to denote z_q originally satisfies the inequality condition $1 \leq \gamma \leq \frac{\|z_p - z_{p-1}\|^2}{\|z_{p+1} - z_{p-1}\|^2}$, and the range of γ_{2k} is set to satisfy (57) for convergence.

Given the designed sequence $\{Z_k\}_{k\geq 0}$, an accelerated ILC scheme can be implemented as follows:

$$u_{k+1} = \mathbf{u}_k + G^*(I + GG^*)^{-1} e_k, \tag{62}$$

$$\mathbf{u}_{k+1} = \mathbf{u}_k + \gamma_{k+1} \left(u_{k+1} - \mathbf{u}_k \right), \tag{63}$$

where $\{u_k\}_{k\geq 0}$ is the actual control input sequence and $\{\mathbf{u}_k\}_{k\geq 0}$ is an auxiliary sequence with $\mathbf{u}_0 = u_0$, and

$$1 \leq \gamma_{k+1} \leq \frac{\|e_{k+1}\|_Q^2 + \|u_{k+1} - \mathbf{u}_k\|_R^2}{\|u_{k+1} - \mathbf{u}_k\|_R^2}. \tag{64}$$

The accelerated scheme in this paper is specifically developed for the nonidentical trial length problem, and the convergence analysis is conducted using framework of alternating projections between multiple closed subspaces. In this sense, by Theorem 3, the accelerated ILC scheme also achieves its convergence properties under alternating projections as in the next result.

Theorem 4. *Given a system described by (1), suppose that the accelerated ILC scheme (62), (63) and (64) is applied with initial input signal u_0. Then*

$$\lim_{k\to\infty} u_{k+1} = U_\infty, \quad \lim_{k\to\infty} \|e_k\| = 0, \tag{65}$$

where $Z_\infty = (0, U_\infty)$, and

$$\|u_k - U_\infty\| \geq \|u_{k+1} - U_\infty\|. \tag{66}$$

Proof. Firstly, the pairs of error and input signals (e_{k+1}, u_{k+1}) and $(0, \mathbf{u}_k)$ in the accelerated ILC scheme (62) and (63) represent Z_{2k+1} and Z_{2k} in the sequence $\{Z_k\}_{k\geq 0}$

as shown in Fig. 3. By Theorem 3, the sub-sequence $\{Z_{2k+1}\}_{k\geq 0}$ also converges to Z_∞, and hence

$$\lim_{k\to\infty} \|Z_{2k+1} - Z_\infty\| = \lim_{k\to\infty} \left\{ \|e_{k+1} - 0\|_Q^2 + \|u_{k+1} - U_\infty\|_R^2 \right\} = 0, \tag{67}$$

which gives (65). Moreover, Z_∞ is also a point in $\bigcap_0^\infty M_j$, then it follows that

$$\begin{aligned} \|Z_{2k+1} - Z_\infty\|^2 &= \|Z_{2k+1} - z_{2k+2}\|^2 + \|z_{2k+2} - Z_\infty\|^2 + 2\langle Z_{2k+1} - z_{2k+2}, z_{2k+2} - Z_\infty\rangle \\ &= \|Z_{2k+1} - z_{2k+2}\|^2 + \|z_{2k+2} - Z_\infty\|^2, \end{aligned} \tag{68}$$

and

$$\begin{aligned} \|Z_{2k} - Z_\infty\|^2 &= \|Z_{2k} - Z_{2k+1}\|^2 + \|Z_{2k+1} - Z_\infty\|^2 + 2\langle Z_{2k} - Z_{2k+1}, Z_{2k+1} - Z_\infty\rangle \\ &= \|Z_{2k} - Z_{2k+1}\|^2 + \|Z_{2k+1} - Z_\infty\|^2. \end{aligned} \tag{69}$$

Combined with (68) and (69), it follows that $\|z_{2k+2} - Z_\infty\|^2 \leq \|Z_{2k} - Z_\infty\|^2$. Also since $\gamma_{2k} \geq 1$, it follows that $\|Z_{2k} - Z_\infty\|^2 \leq \|z_{2k} - Z_\infty\|^2$, and hence

$$\|z_{2k+2} - Z_\infty\|^2 \leq \|z_{2k} - Z_\infty\|^2. \tag{70}$$

Moreover, z_{2k} and z_{2k+2} corresponds to $(0, u_k)$ and $(0, u_{k+1})$, respectively. Hence (66) can be obtained from (70) and the proof is complete. □

Theorem 4 gives the convergence properties of the accelerated ILC scheme (62) and (63) for γ_{k+1} satisfying (64) under alternating projections. Moreover, a more precise selection can be made for some specific convergence properties. A selection for the accelerated scheme is given next.

To start, two auxiliary points $Z'_{2k} = (\mathbf{e}_k, \mathbf{u}_k)$ and $Z'_{2k+2} = (\mathbf{e}_{k+1}, \mathbf{u}_{k+1})$, both of which belong to $M_{j_{2k+1}}$ as shown in Fig. 3, are defined to explore the accelerated property in the case when the trial lengths are not identical. Since $Z'_{2k} \in M_{j_{2k+1}}$, it follows that $\langle Z_{2k} - Z_{2k+1}, Z_{2k+1} - Z'_{2k}\rangle = 0$. Moreover,

$$\|Z'_{2k} - Z_{2k}\|^2 = \|Z_{2k} - Z_{2k+1}\|^2 + \|Z_{2k+1} - Z'_{2k}\|^2, \tag{71}$$

and hence

$$\|\mathbf{e}_k\|_Q^2 = \|Z_{2k} - Z_{2k+1}\|^2 + \|e_{k+1} - \mathbf{e}_k\|_Q^2 + \|u_{k+1} - \mathbf{u}_k\|_R^2. \tag{72}$$

Similarly, since $\left\langle Z_{2k} - Z_{2k+1}, Z_{2k+1} - Z'_{2k+2}\right\rangle = 0$,

$$\|\mathbf{e}_{k+1}\|_Q^2 + \|\mathbf{u}_{k+1} - \mathbf{u}_k\|_R^2 = \|Z_{2k} - Z_{2k+1}\|^2 + \|e_{k+1} - \mathbf{e}_{k+1}\|_Q^2 + \|u_{k+1} - \mathbf{u}_{k+1}\|_R^2. \tag{73}$$

Since both Z'_{2k} and Z'_{2k+2} belong to $M_{j_{2k+1}}$ and combining with (63), (72) and (73), it follows that

$$\|\mathbf{e}_{k+1}\|_Q^2 - \|\mathbf{e}_k\|_Q^2 = \left(\gamma_{k+1}^2 - 2\gamma_{k+1}\right)\|e_{k+1} - \mathbf{e}_k\|_Q^2 - 2\gamma_{k+1}\|u_{k+1} - \mathbf{u}_k\|_R^2. \tag{74}$$

The right-hand side of (74) is a quadratic function with respect to γ_{k+1}, and hence the extreme point, $\gamma_{k+1} = 1 + \frac{\|u_{k+1}-\mathbf{u}_k\|_R^2}{\|e_{k+1}-\mathbf{e}_k\|_Q^2}$, can be selected to minimize the norm of auxiliary error $\mathbf{e}_{k+1}$. For practical implementation, this extreme point cannot always satisfy (64), and in this case, set

$$\gamma_{k+1} = \begin{cases} 1 + \frac{\|u_{k+1}-\mathbf{u}_k\|_R^2}{\|e_{k+1}-\mathbf{e}_k\|_Q^2}, & \text{(64) is satisfied,} \\ 1, & \text{otherwise.} \end{cases} \tag{75}$$

Compared with the accelerated scheme for the constant trial length case, the accelerated sequence $\{Z_k\}_{k\geq 0}$ cannot guarantee the norm of actual error signal is monotonically decreasing by selecting the value of γ_{k+1}. However, the developed accelerated scheme relaxes the limitation that each trial length should be strictly identical. Nonetheless, better performance can be still achieved by following (75), which will be verified in the numerical case study of the next section.

5. Numerical simulation

The example used is described by the following linear continuous MIMO system from [17]

$$\begin{aligned} \dot{x}_k(t) &= \begin{bmatrix} 0.14 & 0.12 & 0.26 & 0.6 \\ -0.3 & 1.2 & 0.23 & -0.3 \\ 0.18 & -0.24 & -0.35 & 0.43 \\ 0.2 & 0.01 & 0.05 & -0.5 \end{bmatrix} x_k(t) + \begin{bmatrix} 0.1 & 0.2 \\ -1 & 8 \\ 1.5 & 9 \\ 0 & 0 \end{bmatrix} u_k(t), \\ y_k(t) &= \begin{bmatrix} 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x_k(t). \end{aligned} \tag{76}$$

The reference trajectory is chosen as

$$\begin{aligned} y_d^{(1)}(t) &= 1.6t^2\left[1 + \cos(\pi \mathrm{t} - \pi)\right], \\ y_d^{(2)}(t) &= 0.7\left[1 + \sin\left(2\pi \mathrm{t} - \frac{\pi}{2}\right)\right], \end{aligned} \tag{77}$$

where the superscript $(\cdot)$ denotes the component of MIMO systems and the desired time duration is $t \in [0,\ 2]$, and hence $T = 2$s. The variation of trial lengths is uniformly distributed, starting from 1.8s, i.e., $T_- = 1.8$s.

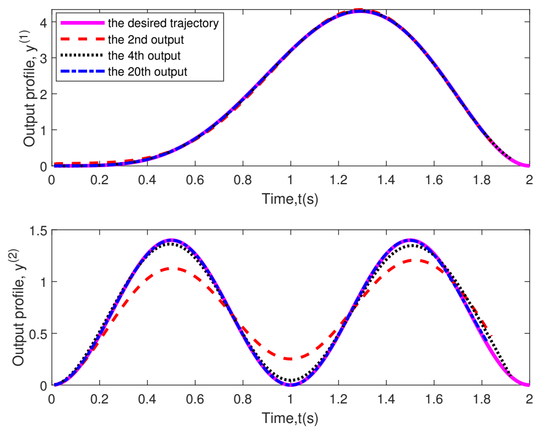

Figure 4: Output profiles of the modified norm optimal ILC.

The alternating projections can be implemented by the norm optimal ILC (40) with modifications, so the simple feedforward form of the modified norm optimal ILC is employed to minimize the cost function (39). Choose the cost function weighing matrices as $Q = 100I_Q$ and $R = 0.001I_R$, where $I_Q = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Set the initial input signal u_0 as 0. Then, simulation with 20 trials gives the results in Fig. 4, where the 2nd and 4th output profiles are specifically plotted and their trial lengths all fail to reach to the desired one. Nonetheless, the 20th output can still track the desired trajectory. The variation profile of the trial length T_k is given in Fig. 5. The cost function defined in (39) is plotted in Fig. 6, which decreases along the trial axis. To further assess performance, the root mean square error (RMSE) of each trial is used. The calculation of RMSE is as follows:

$$RMSE = \sqrt{\frac{1}{T_k} \int_0^T e_k^T(t)\, e_k(t)\, dt}. \tag{78}$$

Comparisons with both the P-type ILC using Arimoto-like gain and the trial averaging ILC [38] are given in Fig. 7. Both learning gain parameters for these alternative designs are set as 0.5 and satisfy the respective convergence conditions for this example. It can be seen in Fig. 7 that the convergence performance of the modified norm optimal ILC is better than that of the other two non-optimized schemes.

An obvious step in this study is to examine the effects of choosing different weighed matrices. In Fig. 8, the RMSEs of the modified norm optimal ILC with

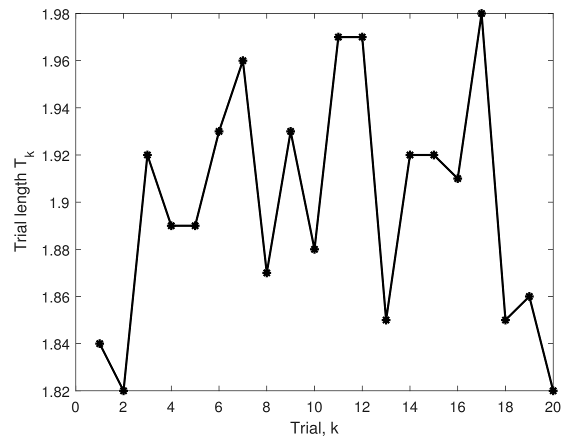

Figure 5: Variation profile of the trial length T_k along the trial.

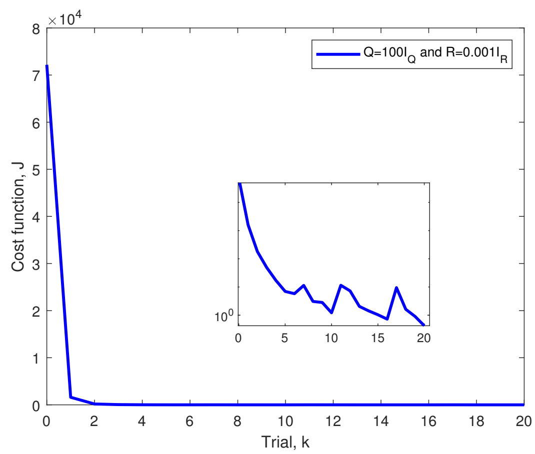

Figure 6: Cost function (39) of the modified norm optimal ILC.

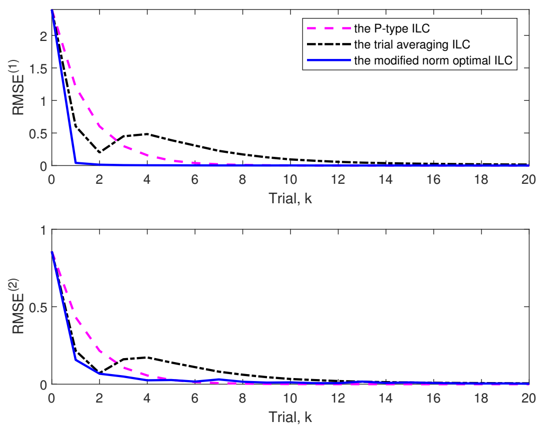

Figure 7: RMSEs of the modified norm optimal ILC compared with the conventional counterparts.

different choices of the weighing matrices are given. Note that both increasing the diagonal entry of Q and decreasing R will have positive effects on the convergence speed. Also, the error convergence will follow the same profile if there exists the same ratio, even though the actual values of the weighing matrices are different. The results can be interpreted under the framework of alternating projections. Different choices of the weighing matrices will change the angle of M_j and M_0, which naturally influences the optimization scale. If the ratios of two choices are same, the angle will also remain unchanged and the error convergence will be the same (under the same variation profile of T_k). Besides, the error convergence is not monotonic, which has been discussed at the end of Section 3. For comparisons, the specific monotonic convergence result in Proposition 2 is given in Fig. 9, where the chosen weighing matrices satisfy the condition (51).

Finally, the effectiveness of the accelerated scheme is examined with $Q = 10I_Q$ and $R = 0.001I_R$ and also γ_{k+1} is chosen using (75). The simulation is run with 20 trials, which generates the data for the modified norm optimal ILC and its accelerated scheme shown in Fig. 10. These confirm that the accelerated scheme indeed can increase the convergence speed. Note that since the initial value of auxiliary sequence $\mathbf{u}_0$ is set as u_0, so $\gamma_1 = 1$ according to (75), which leads to the first trial decrease of the modified norm optimal ILC is same with its accelerated scheme. Other reasonable value of $\mathbf{u}_0$ can be chosen for larger decrease. Also, the profiles of the accelerated scheme have more fluctuations than that of its original method. This is because the accelerated scheme converges faster to z_∞ and the deviation in norm of trial

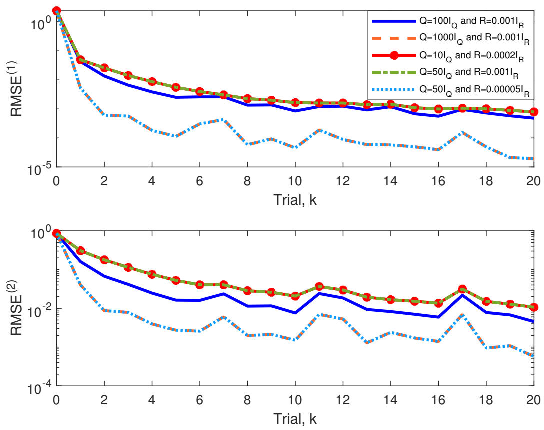

Figure 8: RMSEs of the modified norm optimal ILC with different choices of Q and R.

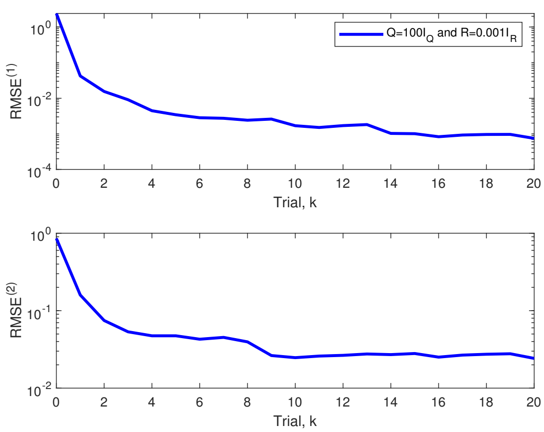

Figure 9: Monotonic convergence of $\bar{e}_k$ in Proposition 2.

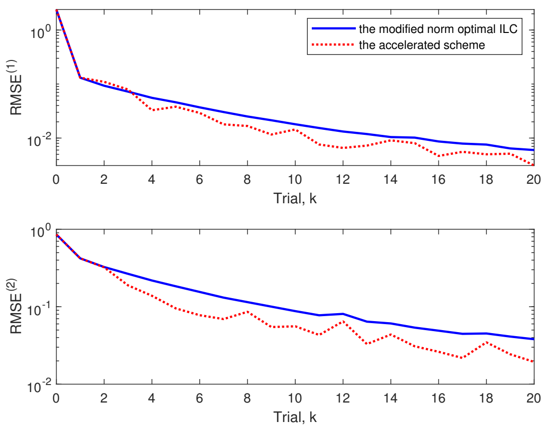

Figure 10: RMSEs of the modified norm optimal ILC and its accelerated scheme with $Q = 10I_Q$ and $Q = 0.001I_R$.

lengths between two successive trials are more likely to be larger on the scale than the distance that is decreasing. Finally, Fig. 11 gives the value of γ along the trial.

6. Conclusions and future work

This paper has developed an optimal ILC design for linear continuous-time MIMO systems with nonidentical trial lengths using alternating projections. By transforming the ILC design to one finding a point inside the intersection region of multiple closed affine subspaces, a projection sequence was developed by defining a projecting order between a single subspace and a family of affine subspaces. The projection sequence was then proved to converge in norm based on an assumption that there exists a point in the intersection region. It was also proved that the designed projection sequence can be implemented by the norm optimal ILC with specific modifications for the nonidentical trial lengths.

An accelerated scheme has also been developed under alternating projections. The accelerated scheme can also be abstracted as a sequence in the defined Hilbert space, which was similarly proved to converge in norm. Moreover, an accelerated implementation for nonidentical trial length problem has been developed based on the modified norm optimal ILC and the choice of the accelerated factor is given. Finally, the effectiveness of the ILC designs has been compared with two other non-optimized ILC designs.

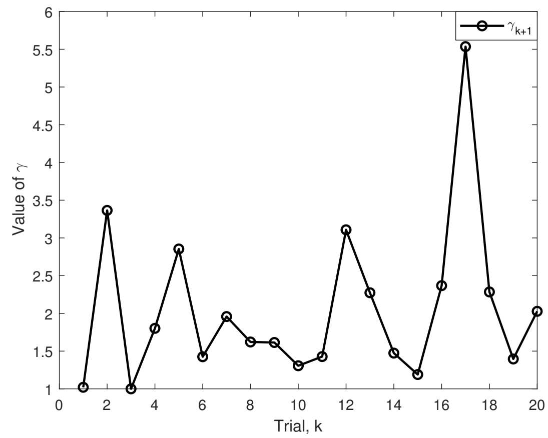

Figure 11: Value change of the accelerated factor γ.

Possible onward developments for future study, include extension to other ILC problems, such as varying initial conditions and nonidentical time scales.

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